Towards an Internal Construction of Meaning

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The concept of mathematical proof

According to Leibniz (~ 1690)

Necessary truths are identical, some explicitly – these are the first truths or axioms – others "virtually" or implicitly, and these are the demonstrable theorems. To demonstrate the latter is to reduce them to identical truths by analyzing their terms, that is, by defining them. Every demonstration consists in substituting the definition for the defined, that is, in replacing a (complex) term by a group of (simpler) terms that is equivalent to it. Thus, the essential foundation of deduction is the principle of the substitution of equivalents. This is the supreme and unique principle of Logic, and not the principle of the syllogism, for the latter [...] is a theorem that is itself proved by means of the former principle.[1]

[1] Couturat, *La Logique de Leibniz*.

Primitive concepts need further elucidation

According to Gödel (1944)

> Many symptoms show only too clearly, however, that the primitive concepts need further elucidation.
> It seems reasonable to suspect that it is this incomplete understanding of the foundations which is responsible for the fact that Mathematical Logic has up to now remained so far behind the high expectations of Peano and others who (in accordance with Leibniz's claims) had hoped that it would facilitate theoretical mathematics to the same extent as the decimal system of numbers has facilitated numerical computations.
> ...
> But there is no need to give up hope. Leibniz did not in his writings about the *Characteristica Universalis* speak of a Utopian project[3]

[3] Gödel, "Russell's Mathematical Logic".

Primitive concepts need further elucidation

According to Gödel (1944)

The very existence of the concept of, e.g., 'class' constitutes already such an axiom; since, if one defined, e.g., 'class' and 'ϵ' to be 'the concepts satisfying the axioms', one would be unable to prove their existence.[2]

[2] Gödel, "Russell's Mathematical Logic".

Is the distinction between syntax and semantics necessary?

- **Axiom vs Definition:**
  - One can define a concept as 'the concept satisfying the axioms'
  - An ambiguity still remains because of the undefined terms
- **Main Thesis:** The distinction between syntax and semantics
  - is responsible of some foundational issues
  - yields redundancies and complications
  - is a consequence of the lack of definition of certain 'concepts'
- **Method:**
  - Organise the objects into a hierarchy of nested n-types / n-categories
  - Internalise typing judgments by expressing them as truth values, i.e. as elements of the level -1 of the hierarchy.
  - Address the junction between syntax and semantics through the act of definition.

Internal Type Constructors

	Syntactic world		Semantic world
$(\_ : X)$:	$\mathbb{L}$	$\longrightarrow$	$\mathbb{B}$

- $\mathbb{L}$ is the (po)set of symbols that we use to write. It is constructed dynamically:
  - $x := y \Rightarrow \mathbb{L}(x, y)$
- $\mathbb{B}$ is the poset of truth values
- $(\_ : X)$ maps a symbol x to a truth value $(x : X)$
- $(x : X)$ gives conditions for a symbol x to be defined as an element of X.
  - We can work formally by deriving the logical consequences of $(x : X)$
  - An element of X is a symbol x equipped with a proof $def_x : (x : X)$

The fundamental constructors

An inductive construction of n-categories and n-functors

$$
(X : \textsc{Cat}_n) \quad \underset{\text{Synthesis}}{\overset{\text{Analysis}}{\rightleftarrows}} \quad := \quad \begin{array}{lrcl} - & (\_ : X) & : & \textsc{Cat}_0\,(\mathbb{L}, \mathbb{B}) \\ - & X(\_, \_) & : & \textsc{Cat}_n\,(X^{\mathrm{op}} \times X, \textsc{Cat}_{n-1}) \end{array}
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F : \text{CAT}_n(Y, Z) \quad := \quad \begin{array}{lrcl} - & (y : Y) & \Rightarrow & (Fy : Z) \\ - & F(\_, \_) & : & \int^{x,y : Y^{\text{op}} \times Y} \text{CAT}_{n-1}\,(Y(x,y), Z(Fx, Fy)) \end{array}
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The fundamental constructors

An inductive construction of *n*-categories and *n*-functors

$$(X : \mathrm{CAT}_n) \quad := \quad \begin{array}{rrcl} - & (\_ : X) & : & \mathrm{CAT}_0\,(\mathbb{L}, \mathbb{B}) \\ - & X(\_, \_) & : & \mathrm{CAT}_n\,(X^{\mathrm{op}} \times X, \mathrm{CAT}_{n-1}) \end{array}$$

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This definition acquires a meaning as soon as $(\mathrm{CAT}_{n-1} : \mathrm{CAT}_n)$ has been proven

$$(\mathrm{CAT}_{n-1} : \mathrm{CAT}_n) \quad := \quad \begin{array}{rrcl} - & (\_ : \mathrm{CAT}_{n-1}) & : & \mathrm{CAT}_0\,(\mathbb{L}, \mathbb{B}) \\ - & \mathrm{CAT}_{n-1}(\_, \_) & : & \mathrm{CAT}_n\,(\mathrm{CAT}_{n-1}^{\mathrm{op}} \times \mathrm{CAT}_{n-1}, \mathrm{CAT}_{n-1}) \end{array}$$

The Definition Process

Assignment, literal equality, and substitution

- We regard the act of definition as a process that introduces a literal equality between symbols
  - Writing $x := y$ for $x, y : \mathbb{L}$ updates the (po)set of symbols $\mathbb{L}$
  - $x := y$ should be seen as a proof of the literal equality $\mathbb{L}(x, y)$
- We obtain from the definition of morphisms that a proof of

$$(\_ : X) : \textsc{Cat}_0(\mathbb{L}, \mathbb{B})$$

  corresponds to:
  - $x : \mathbb{L} \Rightarrow (x : X) : \mathbb{B}$
  - $x, y : \mathbb{L} \Rightarrow (\mathbb{L}(x, y) \Rightarrow \mathbb{B}((x : X), (y : X)))$
- In particular, $x := y \Rightarrow \mathbb{B}((x : X), (y : X))$

The meaning of logical rules is to be found in the rules themselves

According to Girard, *The meaning of logical rules* (1998)

- The definition obtained for truth values is as follows:

$$\begin{array}{rclcrcl} (\tau : \textsc{Cat}_{-1}) & := & - & (\_ : \tau) & : & \mathbb{L} \to \mathbb{B} \\ & & - & \tau(\_, \_) & : & \tau^{\mathrm{op}} \times \tau \to \textsc{Cat}_{-1-1} \end{array}$$

- The heart of the internalisation of meaning lies in the identification of $\mathbb{B}$ with $\textsc{Cat}_{-1}$.
- There's some hardcoding to be done here: we first fix some notations and then check that the poset truth values fits within the hierarchy

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What is Truth?

True is what has a proof

- We set:

$$(\alpha : \top) := \top$$

- This provides a semantic equivalence, i.e. an equivalence within $\mathbb{B}$:

$$(\alpha : \top) \Leftrightarrow \top$$

- **It can not be expressed as such in a type theory.**

- The proof of *true*, α, is to be seen as the smallest semantic unit here.
- Here mathematical truth is constructed by definitions only.

Bishop's sets

1985

A set is not an entity which has an ideal existence. A set exists only when it has been defined.
To define a set we prescribe, at least implicitly, what we (the constructing intelligence) must do in order to construct an element of the set, and what we must do to show that two elements of the set are equal.
A similar remark applies to the definition of a function: in order to define a function from a set A to a set B, we prescribe a finite routine which leads from an element of A to an element of B, and show that equal elements of A give rise to equal elements of B.[4]

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[7] Bishop, *Constructive Analysis*.

Bishop's sets

An *n*-categorical extension

An *n*-category is not an entity which has an ideal existence. An *n*-category exists only when it has been defined.

To define an *n*-category we prescribe, ~~at least implicitly~~, what we (the constructing intelligence) must do in order to construct an object of the *n*-category, and what we must do to construct the $n-1$-category of morphisms between each pair of objects *in a natural way*.

A similar remark applies to the definition of an *n*-functor: in order to define an *n*-functor from an *n*-category A to an *n*-category B, we prescribe a finite routine which leads from an object of A to an object of B, and construct an $n-1$-functor from the $n-1$-category of morphisms between each pair of objects of A to the $n-1$-category of morphisms between the image pair of objects of B, *in a natural way*.

A Toy Example

Constructing the Empty Set

- We prove that there is a set with no element ($\varnothing : \mathrm{SET}$).
  - In type theory, there is no way to construct an element of $\varnothing$.
  - Here, being an element of $\varnothing$ is wrong by definition.

- $(\_ : \varnothing) : \mathbb{L} \to \mathbb{B}$
  - $(x : \mathbb{L}) \Rightarrow (x : \varnothing) := \bot$
  - $(x, y : \mathbb{L}) \Rightarrow (\mathbb{L}(x, y) \Rightarrow \mathbb{B}((x : \varnothing), (y : \varnothing)))$
- $\varnothing(\_, \_) : \varnothing^{\mathrm{op}} \times \varnothing \to \mathbb{B}$
  - $(x, y : \varnothing) \Rightarrow \varnothing(x, y) := \top$
  - $(x_1, x_2, y_1, y_2 : \varnothing) \Rightarrow (\varnothing^{\mathrm{op}}(x_1, y_1) \times \varnothing(x_2, y_2) \Rightarrow \mathbb{B}(\varnothing(x_1, x_2), \varnothing(y_1, y_2)))$

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Godel's Sentence and The Liar Paradox

both correspond to the same undefinable sentence here

- Because *true* is defined as *provable* and $\mathbb{B}(\top, \bot) = \bot$, the statements
  - *I am not provable*
  - *This sentence is false*

  both correspond to a fixed point S for the negation

$$\mathbb{B}(\_, \bot) : \mathbb{B}^{\mathrm{op}} \to \mathbb{B}$$

  that is: $(S : \mathbb{B}) \times (S \Leftrightarrow \mathbb{B}(S, \bot))$
- **N.B.** The diagonal argument can not be used because of the contravariance of the negation
- In Boolean logic, $(S : \mathbb{B}) \Leftrightarrow (S \Leftrightarrow \bot) \sqcup (S \Leftrightarrow \top)$. We obtain
- **N.B.** It remains envisionable to construct a framework where a truth value is equivalent to its own negation.

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Universes and Size Issues

A Hope

- Classical set-theoretic results do not *a priori* hold in this framework.
- Explicitly specifying the equality relation between elements yield significant distinctions with ZFC's set theory
- **Ex:** Cantor-Bernstein's theorem

$$(X, Y : \textsc{Set}) \Rightarrow ((X < Y) \times (Y < X) \Rightarrow (X \simeq Y))$$

  might no longer hold in this framework
  - At least two of the proofs do no longer hold
  - One proof involves the construction of a subset as a limit of subsets
  - But in this framework, the modified equality yields identifications on the elements of that limit
  - Hence the resulting set is a quotient of a subset and not a subset anymore
- Quotients behave differently
- Objects of an n-category are not discernable as standalone entities for $n > 0$

Universes and Size Issues

Rethinking the role and the structure of the gradation

- The hope is that there is no threat against the existence of a functor

$$\mathrm{S{\scriptsize ET}}(\_,\_) : \mathrm{S{\scriptsize ET}}^{\mathrm{op}} \times \mathrm{S{\scriptsize ET}} \to \mathrm{S{\scriptsize ET}} \qquad \text{in } \mathrm{C{\scriptsize AT}}$$

- and more generally,

$$\mathrm{C{\scriptsize AT}}_n(\_,\_) : \mathrm{C{\scriptsize AT}}_n^{\mathrm{op}} \times \mathrm{C{\scriptsize AT}}_n \to \mathrm{C{\scriptsize AT}}_n \qquad \text{in } \mathrm{C{\scriptsize AT}}_{n+1}$$

- The notion of size is addressed internally by using explicit inter-dimensional connections – the following inductively defined adjunction within $\mathrm{C{\scriptsize AT}}_{n+1}$:

$$\Pi_{n-1} \quad : \quad \mathrm{C{\scriptsize AT}}_n \quad \rightleftharpoons \quad \mathrm{I}_n\mathrm{C{\scriptsize AT}}_{n-1} \quad : \quad \mathrm{I}_n$$

Universes and Size Issues

Rethinking the role and the structure of the gradation

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$$\mathrm{SET}(\_,\_) : \mathrm{CAT}\left(\mathrm{SET}^{\mathrm{op}} \times \mathrm{SET}, \mathrm{SET}\right)$$

- and more generally,

$$\mathrm{CAT}_{n}(\_,\_) : \mathrm{CAT}_{n+1}\left(\mathrm{CAT}_{n}^{\mathrm{op}} \times \mathrm{CAT}_{n}, \mathrm{CAT}_{n}\right)$$

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