Building All of Mathematics without Axioms

An n-Categorical Manifesto

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March 19, 2025

Building All of Mathematics without Axioms

An n-Categorical Manifesto

PhD Thesis work

- *Operads in 2-categories and models of structure interchange*
- Initiated by the need to
  - manage constructions involving large categories
  - describe more complex structures and symmetries

  to understand the structure of n-fold iterated loop spaces.

Idea

- Define everything.
- Internalize type constructors
- Allow internal reasoning only.
- Capture all of dependencies through functoriality.

Timeline

Foundations of Logic and Mathematics

- **Maths written in natural language.**
- Axioms as proof-irrelevant truth.

-350	*Aristotle*	Logic
-300	*Euclide*	Axiomatization of Mathematics
1666	*Leibnitz*	Let's prove the axioms!

- **Formalization of mathematical language.**
- Boundary between axiom and definition fades away.

1847	*Boole*	Logic
1883	*Cantor*	Set Theory
1894	*Peano*	Symbolic Language, Natural Numbers
1898	*Hilbert*	Formal Systems

- **Emergence of mathematical logic**.
- Truth as deducibility within a coherent formal system.

1901	Russel's Paradox
1922	ZFC
1931	Gödel's incompleteness, Tarski's undefinability of truth
1942	Category Theory

Formal Systems

The Idea

- A formal system enables to deduce truth about things from simple truth about those things.
- Formally assume that
    - some statements hold (axioms)
    - some statements can be deduced from other (rules)
- Combine the rules to obtain syntactic consequences
    - Write $K \vdash J$ if J can be derived from K (proof tree)
    - e.g. $\dfrac{\dfrac{J_1^1 \cdots J_1^{n_1}}{J_1} \cdots \dfrac{J_p^1 \cdots J_p^{n_p}}{J_p}}{J} \Rightarrow J_1^1 \cdots J_p^{n_p} \vdash J$
- The results are valid for each object satisfying the axioms.

Meta Level		Model	F S	Expected truth value
0	Things	Objects	T, x	
1	Judgments	Statements	J	$\vdash J$
2	Rules	Hold	$\frac{J_1 \ldots J_r}{J}$	$(\prod_i \vdash J_i) \Rightarrow (\vdash J)$
2	Axioms	Hold	$\overline{A}$	$\top \Rightarrow (\vdash A)$
3	Entailment	Hold	$K \vdash J$	$(\vdash K) \Rightarrow (\vdash J)$

Formal Systems

Entailment

Ultimately, we care about the derivability of judgments. Form a poset $(\mathbb{J}, \vdash)$

- whose objects are concatenation of judgments: $\varnothing, J_1 \cdots J_r$
- whose order is generated by the rules in a way compatible with concatenation, so that
  - $\mathbb{J}(J_1 \ldots J_r, J) = J_1 \ldots J_r \vdash J$
  - $\prod_{i=1}^{r} \mathbb{J}\left(J_i^\bullet, J_i\right) \Rightarrow \mathbb{J}\left(J_1^\bullet \cdots J_r^\bullet, J_1 \cdots J_r\right)$ $\qquad \dfrac{J_1^1 \cdots J_1^{n_1}}{J_1} \cdots \dfrac{J_r^1 \cdots J_r^{n_r}}{J_r}$
- Human interpretation of a deductive system:

$$\begin{array}{lcccc} \mathbb{J}(\varnothing, \_) : & (\mathbb{J}, \vdash) & \xrightarrow{\vdash \_} & (\mathbb{B}, \Rightarrow) & \\ & J & \mapsto & \vdash J & \\ & K \vdash L & \Rightarrow & (\vdash K \Rightarrow \vdash L) & \qquad \dfrac{K}{L} \Rightarrow \left(\dfrac{\varnothing}{K} \Rightarrow \dfrac{\frac{\varnothing}{K}}{L} \right) \end{array}$$

$$(\vdash \varnothing) = \top$$
$$(\vdash J_1 \cdots J_r) = (\vdash J_1) \times \cdots \times (\vdash J_r)$$

$$\frac{J_1 \cdots J_r}{J} \Rightarrow (\vdash J_1) \times \cdots \times (\vdash J_r) \Rightarrow (\vdash J)$$

A Significant Problem

Deductive Systems and Boolean Logic

While the deductive process is confined to Boolean logic, it fails to benefit from its essential reasoning features.

	Deductive System		**Real Life**
$\vdash \_:$	$(\mathbb{J}, \vdash)$	$\longrightarrow$	$(\mathbb{B}, \Rightarrow)$
	∘ No structural $\bot$		∘ Contradiction $(\vdash J) \times (\vdash J \Rightarrow \bot) \Rightarrow \bot$
	∘ $(\vdash J \Rightarrow \bot)$ doesn't give $J \vdash K$ for all K		∘ Ex falso quod libet $\bot \Rightarrow \vdash K$
	∘ Can't derive $J \vdash K$ from $\vdash J \Rightarrow \vdash K$		∘ Proof-based implication $(\top \Rightarrow \vdash J) \Rightarrow (\top \Rightarrow \vdash K) \Leftrightarrow (\vdash J \Rightarrow \vdash K)$

Consequence: Some truth are send outside of the system:

$$\mathbb{J}(\varnothing, \_) : \mathbb{J}(J, K) \to \mathbb{B}(\vdash J, \vdash K)$$

Formal Systems vs Internal Reasoning

Formal Systems Suffer from their External Interpretation

Notation	**What It Is**	**What It Should Be**
$(\mathbb{J}, \vdash)$	A Cartesian Poset $\_ \vdash \_ : \mathbb{J}^{op} \times \mathbb{J} \to \mathbb{B}$	An Heyting Algebra Object $\_ \vdash \_ : \mathbb{J}^{op} \times \mathbb{J} \to \mathbb{J}$
$\vdash J$	A Truth Value $J : \mathbb{J} \Rightarrow \vdash J : \mathbb{B}$ $\varnothing \vdash \_ : \mathbb{J} \to \mathbb{B}$	The Same Judgment $J : \mathbb{J} \Rightarrow (\top \vdash J) \simeq J : \mathbb{J}$ $\top \vdash \_ : \mathbb{J} \xrightarrow{Id} \mathbb{J}$
$\varnothing$ or $\top$	The Empty Judgment $(J \vdash \varnothing)$	The Terminal Object $(J \vdash \top) \simeq \top : \mathbb{J}$
$\bot$	It Can Be Introduced $(\bot \vdash J)$?	The Initial Object $(\bot \vdash J) \simeq \top : \mathbb{J}$
$J^{\perp}$	It Can Be Defined e.g. $J^{\perp} := (J \to \bot) : \mathbb{J}$	Structural Negation $\_ \vdash \bot : \mathbb{J}^{op} \to \mathbb{J}$ $(J \vdash K) \vdash (K^{\perp} \vdash J^{\perp})$
$(\vdash J)^{\perp}$ $\Leftrightarrow \vdash J^{\perp}$	?	True $(\vdash J)^{\perp} \Leftrightarrow J^{\perp} \Leftrightarrow \vdash J^{\perp}$

Formal Systems vs Internal Reasoning

Formal Systems Suffer from the External Interpretation

Notation	What It Is	What It Should Be
$(\mathbb{J}, \vdash)$	A Cartesian Poset $\_ \vdash \_ : \mathbb{J}^{\mathrm{op}} \times \mathbb{J} \to \mathbb{B}$	An Heyting Algebra Object $\_ \vdash \_ : \mathbb{J}^{\mathrm{op}} \times \mathbb{J} \to \mathbb{J}$
$\vdash J \times \vdash J^{\perp}$	A Truth Value *consistence*	Structurally False $(\vdash J) \times (J \vdash \perp) \vdash \perp$
$J \vdash K$	A Truth Value *derivable rules*	A Judgment $J \vdash K \ : \mathbb{J}$
$\vdash J \Rightarrow \vdash K$	Another Truth Value *admissible rules*	The Same Judgment $(\vdash J) \vdash (\vdash K) \simeq J \vdash K$

NB The notion of fixed point does not make sense for internal negation

$$(\_)^{\perp} : \mathbb{J}^{\mathrm{op}} \to \mathbb{J}$$

Formal Systems: Key Weaknesses

An Invitation to Internal Reasoning

Intrinsic Structural Shortcomings

- No built-in Heyting algebra structure
- Exclusively deductive: lacks hypothetical reasoning
- Admissible rules are externally valid but internally invalid

Syntactic and Structural Overhead

- Proliferation of implication symbols: $—, \vdash, \models, \Rightarrow, \rightarrow, \vdash$
- Tedious formalization

Expressiveness Limitations

- Ultimately reduces to boolean classic logic *(if), then, and*.
- Cannot express intermediate values of derivability

From Denotation to Meaning by Substitution

We want to **assign a meaning to combinations of symbols**.

Expression	:=	Combination of symbols.
Statement	:=	Expression assigned with a **meaning**.
Meaning	:=	Meaningful combination of meaningful expressions.
Meaningful	:=	Assigned with a truth value.
Truth Value	:=	Possible value for the evaluation of a statement.

Meaning assignment

(some expr) := *(some meaningful comb of meaningful exprs)*

Objective

Determine the **truth value** of statements.

Evaluation

The evaluation process stops at truth values:

The value of a truth value is itself.

In this talk, statements are evaluated as either $\bot$ or $\top$.

Truth Values

Conjunction, Implication and Proof

Truth Tables

$\_ \Rightarrow \_$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$

$\_ \times \_$	$\bot$	$\top$
$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$

Also write:

- $\mathbb{B}(x, y) := x \Rightarrow y$
- $x \Leftrightarrow y := (x \Rightarrow y) \times (y \Rightarrow x)$

Note that $\mathbb{B}(x, y_1 \times y_2) \Leftrightarrow \mathbb{B}(x, y_1) \times \mathbb{B}(x, y_2)$

Proof and Truth

- Truth values are identified by their **proof**.
- True has proof.
- False has no proof
- (Maybe has maybe a proof.)

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Conjunction, Implication and Proof

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Proof and Truth

- Truth values are identified by their proof.
- (**True has proof**) **is true**
- (**False has proof**) **is false**

Truth Values

Conjunction, Implication and Proof

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Proof and Truth

- Truth values are identified by their proof.
- (**True** **has** **proof**) $:=$ **true**
- (**False** **has** **proof**) $:=$ **false**

Truth Values

Conjunction, Implication and Proof

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Note that $(\; \mathbb{B}(x, y_1 \times y_2) \Leftrightarrow \mathbb{B}(x, y_1) \times \mathbb{B}(x, y_2) \;) = \top$

Proof and Truth

- Truth values are identified by their proof.
- $(\top$ **has proof** $) \quad := \quad \top$
- $(\bot$ **has proof** $) \quad := \quad \bot$

Truth Values

Conjunction, Implication and Proof

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$\top$	$\bot$	$\top$

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Proof and Truth

- Truth values are identified by their proof.
- $(\gamma : \top\ ) \quad := \quad \top$
- $(\beta : \bot\ ) \quad := \quad \bot$

Truth Values

Conjunction, Implication and Proof

Truth Tables

$\_ \Rightarrow \_$	$\bot$	$\top$
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$\top$	$\bot$	$\top$

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Consequences

- We say that a statement is true if it has a proof.
- Implication is determined by its behaviour on proofs
- A proof of a product consists of a proof of each factors.

Truth Values

Conjunction, Implication and Proof

Truth Tables

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Proof and Truth

- Truth values are identified by their proof.
- $(\gamma : \top) \quad := \quad \top$
- $(\beta : \bot) \quad := \quad \bot$

Consequences

- P *is true* $\Leftrightarrow p : P \Leftrightarrow P \Leftrightarrow \mathbb{B}(\top, P)$
- $\nu : \mathbb{B}(P, Q) \Leftrightarrow (p : P \Rightarrow \nu p : Q)$
- $\nu : P \times Q \Leftrightarrow (\nu_P : P) \times (\nu_Q : Q)$

Internalizing Typing Judgments

Overview of the Strategy

The objects X that we first consider are equipped with a predicate $(\_ : X) : \mathbb{V} \to \mathbb{B}$.

- $(\_ : X)$ maps a variable x to a truth value $(x : X)$, which we regard as "*x is an element of type X*".
- The object $\mathbb{V} \to \mathbb{B}$ is itself equipped with a predicate $\_ : \mathbb{V} \to \mathbb{B}$, which to $P : \mathbb{V}$ associates $P : \mathbb{V} \to \mathbb{B} :=$
  - $(x : \mathbb{V}) \Rightarrow Px : \mathbb{B}$
  - $x, y : \mathbb{V} \Rightarrow ((x =_{\mathbb{V}} y) \Rightarrow Px \Leftrightarrow Py)$
- **By substitution** of P by $(\_ : X)$, we obtain $(\_ : X) : \mathbb{V} \to \mathbb{B} \Leftrightarrow \mathbb{B}$
  - $(x : \mathbb{V}) \Rightarrow (x : X) : \mathbb{B}$
  - $x, y : \mathbb{V} \Rightarrow ((x =_{\mathbb{V}} y) \Rightarrow (x : X) \Leftrightarrow (y : X))$
- This predicate should be seen as a type constructor...
- ... hence a way to construct, or *define* elements of X.
- An element of X is a variable x equipped with a proof $def_x^X : (x : X)$.

The Definition

Objects. $\_ : \textsc{Cat}_n : \mathbb{V} \to \mathbb{B}$

$$(X : \textsc{Cat}_n) := \begin{array}{lccc} - & (\_ : X) & : & \mathbb{V} \to \mathbb{B} \\ - & X(\_, \_) & : & X^{\mathrm{op}} \times X \to \textsc{Cat}_{n-1} \end{array}$$

Interpretation

An n-category X consists of

- **Objects:** A truth value $(x : X)$ depending on a variable symbol $x : \mathbb{V}$,which gives a way for x to be defined as an element of X.
- **Morphisms:** An $(n-1)$-category $X(x, y)$ for each elements $x, y : X$, together with additional structure promoting this mapping to an n-functor $X(\_, \_)$.

Objective

Provide this sentence with a meaning.

First Observations

$$(X : \mathrm{CAT}_n) := \begin{array}{lrcl} - & (\_ : X) & : & \mathbb{V} \to \mathbb{B} \\ - & X(\_,\_) & : & X^{\mathrm{op}} \times X \to \mathrm{CAT}_{n-1} \end{array}$$

Objective

Assign the right hand side with meaning.

Method

- Assign each line with a truth value.
- The result is obtained by conjunction.

Steps

- $Y : \mathrm{CAT}_n \quad \Rightarrow \quad Y^{\mathrm{op}} : \mathrm{CAT}_n$
- $Y, Z : \mathrm{CAT}_n \quad \Rightarrow \quad Y \times Z : \mathrm{CAT}_n$
- $Y, Z : \mathrm{CAT}_n \quad \Rightarrow \quad \mathrm{CAT}_n(Y, Z) : \mathrm{CAT}_n$

  > Proceed by induction to avoid circularity. Then:
  > $\mathrm{CAT}_n(Y, Z) : \mathrm{CAT}_n \Rightarrow \_ : \mathrm{CAT}_n(Y, Z) \ : \mathbb{V} \to \mathbb{B}$

- $\mathrm{CAT}_{n-1} : \mathrm{CAT}_n$

First Observations

$(\_ : \mathrm{CAT}_n) : \mathbb{V} \to \mathbb{B}$

$$
(X : \mathrm{CAT}_n) := \begin{array}{lrcl} - & (\_ : X) & : & \mathrm{CAT}_0(\mathbb{V}, \mathbb{B}) \\ - & X(\_, \_) & : & \mathrm{CAT}_n(X^{\mathrm{op}} \times X, \mathrm{CAT}_{n-1}) \end{array}
$$

Objective

Assign the right hand side with meaning.

Method

- Assign each line with a truth value.
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Steps

- $Y : \mathrm{CAT}_n \quad \Rightarrow \quad Y^{\mathrm{op}} : \mathrm{CAT}_n$
- $Y, Z : \mathrm{CAT}_n \quad \Rightarrow \quad Y \times Z : \mathrm{CAT}_n$
- $Y, Z : \mathrm{CAT}_n \quad \Rightarrow \quad \mathrm{CAT}_n(Y, Z) : \mathrm{CAT}_n$

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- $\mathrm{CAT}_{n-1} : \mathrm{CAT}_n$

Opposite

$$(X^{\mathrm{op}} : \mathrm{CAT}_n) := \begin{array}{lccl} - & (\_ : X^{\mathrm{op}}) & : & \mathbb{V} \to \mathbb{B} \\ - & X^{\mathrm{op}}(\_,\_) & : & X^{\mathrm{opop}} \times X^{\mathrm{op}} \to \mathrm{CAT}_{n-1} \end{array}$$

The opposite X^{op} of $X : \mathrm{CAT}_n$ is constructed inductively as follows:

- $\_ : X^{\mathrm{op}} := \_ : X$
- $x, y : X^{\mathrm{op}} \Leftrightarrow x, y : X \Rightarrow X^{\mathrm{op}}(x, y) := X(y, x) : \mathrm{CAT}_{n-1}$
- Obtain $X^{\mathrm{opop}} \xrightarrow{\simeq} X$ from $X^{\mathrm{opop}}(x, y) = X^{\mathrm{op}}(y, x) = X(x, y)$

$$X^{\mathrm{op}}(\_,\_) : X^{\mathrm{opop}} \times X^{\mathrm{op}} \xrightarrow{\simeq} X \times X^{\mathrm{op}} \xrightarrow{\tau} X^{\mathrm{op}} \times X^{\mathrm{opop}} \xrightarrow{X(\_,\_)} \mathrm{CAT}_{n-1}$$

Remark

The oppositization is made functorial by using higher oppositization functors

$$\_^{op_r} : \mathrm{CAT}_n^{op_{r+1}} \to \mathrm{CAT}_n$$

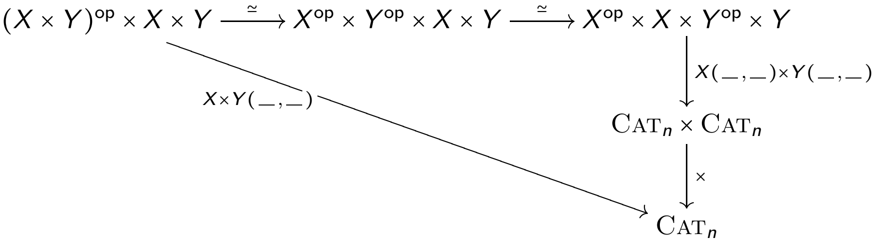
X^{op_r} is such that $x : X^{op_r} \Leftrightarrow x : X$ and $X^{op_r}(x, y) := X(x, y)^{op_{r-1}}$

Products

The product $X \times Y$ of $X, Y : \mathrm{CAT}_n$ is defined inductively as

- $p : X \times Y \quad \Leftrightarrow \quad (p_X : X, p_Y : Y)$
- $p, q : X \times Y \quad \Rightarrow \quad X \times Y(p,q) : \mathrm{CAT}_n$

$$\big|\; X \times Y(p,q) := X(p_X, q_X) \times Y(p_Y, q_Y)$$



By construction, $X \times Y$ is equipped with morphisms in CAT_n

- $\pi_X : X \times Y \to X$
- $\pi_Y : X \times Y \to Y$

Products

By construction, $X \times Y$ is equipped with morphisms in CAT_n

- $\pi_X : X \times Y \to X$
- $\pi_Y : X \times Y \to Y$

The product yields

$$\times : \mathrm{CAT}_n \times \mathrm{CAT}_n \to \mathrm{CAT}_n$$

and satisfies the universal property

$$\mathrm{CAT}_n(\_, X \times Y) \simeq \mathrm{CAT}_n(\_, X) \times \mathrm{CAT}_n(\_, Y)$$

First levels

$\mathbb{B} : \textsc{Cat}_0$

Notation

$\alpha := \textsc{Cat}_{-2-1}$
$\top := \textsc{Cat}_{-1-1}$
$\mathbb{B} := \textsc{Cat}_{0-1}$

- First recall that $\gamma : \mathbb{B} \Rightarrow \mathbb{B}(\gamma, \top) = \top$
- By substitution of n by -1 in $(\tau : \textsc{Cat}_n)$:

$$(\tau : \mathbb{B}) \overset{def}{\Leftrightarrow} \begin{array}{lccc} - & (\_ : \tau) & : & \mathbb{V} \to \mathbb{B} \\ - & \tau(\_, \_) & : & \mathbb{B}\left(\tau^{\text{op}} \times \tau, \top\right) \end{array}$$

First levels

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$$(\tau : \mathbb{B}) \Leftrightarrow \begin{array}{lrcl} - & (\_ : \tau) & : & \mathbb{V} \to \mathbb{B} \\ - & \tau(\_, \_) & : & \top \end{array}$$

First levels

$\mathbb{B} : \textsc{Cat}_0$

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- By substitution of n by -1 in $(\tau : \textsc{Cat}_n)$:

$$(\tau : \mathbb{B}) \;\Leftrightarrow\; (\_ : \tau) : \mathbb{V} \to \mathbb{B}$$

First levels

$\mathbb{B} : \textsc{Cat}_0$

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$\alpha := \textsc{Cat}_{-2-1}$
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$\mathbb{B} := \textsc{Cat}_{0-1}$

- First recall that $\gamma : \mathbb{B} \Rightarrow \mathbb{B}(\gamma, \top) = \top$
- By substitution of n by -1 in $(\tau : \textsc{Cat}_n)$:

$$(\tau : \mathbb{B}) \;\Leftrightarrow\; (\_ : \tau) : \mathbb{V} \to \mathbb{B}$$

- By substitution of n by 0 in $(\mathbb{B} : \textsc{Cat}_n)$:

$$\mathbb{B} : \textsc{Cat}_0 \;\Leftrightarrow\; \begin{array}{lccl} - & (\_ : \mathbb{B}) & : & \mathbb{V} \to \mathbb{B} \\ - & \mathbb{B}(\_, \_) & : & \mathbb{B}^{\mathrm{op}} \times \mathbb{B} \to \mathbb{B} \end{array}$$

$\mathrm{CAT}_n(X, Y) : \mathrm{CAT}_n$

n - Functors

Objective $X, Y \; : \; \mathrm{CAT}_n \; \Rightarrow \; \mathrm{CAT}_n(X, Y) \; : \; \mathrm{CAT}_n$

By Definition We need to show the following:

- $(\_ : \mathrm{CAT}_n(X, Y)) \; : \; \mathbb{V} \to \mathbb{B}$
- $\mathrm{CAT}_n(X, Y)(\_, \_) \; : \; \mathrm{CAT}_n(X, Y)^{\mathrm{op}} \times \mathrm{CAT}_n(X, Y) \to \mathrm{CAT}_{n-1}$

First Step

$F : \mathrm{CAT}_n(X, Y) :=$

- $x : X \Rightarrow Fx : Y$
- $F(\_, \_) : \int^{X^{\mathrm{op}} \times X} \mathrm{CAT}_{n-1}\left(X(x, y), Y(Fx, Fy)\right)$

The end is defined by universal property and encodes naturality. Its elements can be constructed inductively

- $x, y : X \quad \Rightarrow \quad F(x, y) : X(x, y) \to Y(Fx, Fy)$
- $x_1, x_2, y_1, y_2 : X \quad \Rightarrow \quad F^{(2)}_{x_1, x_2, y_1, y_2} : \int^{f_x : X^{\mathrm{op}}(x_1, x_2), f_y : X(y_1, y_2)} \ldots$

$\mathrm{C{\scriptsize AT}}_n(X, Y) : \mathrm{C{\scriptsize AT}}_n$

n - Functors

Objective $\quad X, Y \;:\; \mathrm{C{\scriptsize AT}}_n \;\Rightarrow\; \mathrm{C{\scriptsize AT}}_n(X, Y) \;:\; \mathrm{C{\scriptsize AT}}_n$

First Step

$F : \mathrm{C{\scriptsize AT}}_n(X, Y) :=$

- $x : X \quad \Rightarrow \quad Fx : Y$
- $F(\_, \_) : \int^{X^{\mathrm{op}} \times X} \mathrm{C{\scriptsize AT}}_{n-1}\,(X(x, y), Y(Fx, Fy))$

The end is defined by universal property and encodes naturality.
Its elements can be constructed inductively

$$\int^{X^{\mathrm{op}} \times X} \mathrm{C{\scriptsize AT}}_{n-1}\,(X(x, y), Y(Fx, Fy))$$

$$\simeq [X^{*\mathrm{op}} \times X^{*}, \mathrm{C{\scriptsize AT}}_{n-1}]\,(X^{*}(\_, \_), \mathrm{C{\scriptsize AT}}_{n-1}\,(X(\_, \_), Y(F\_, F\_)))$$

$$\simeq \int^{X^{*\mathrm{op}} \times X^{*}} \mathrm{C{\scriptsize AT}}_{n-1}\,(X^{*}(\_, \_), \mathrm{C{\scriptsize AT}}_{n-1}\,(X(\_, \_), Y(F\_, F\_)))$$

$\mathrm{CAT}_n(X,Y) : \mathrm{CAT}_n$

n - Functors

Objective $\quad X, Y \;:\; \mathrm{CAT}_n \;\Rightarrow\; \mathrm{CAT}_n(X,Y) \;:\; \mathrm{CAT}_n$

First Step

$F : \mathrm{CAT}_n(X,Y) :=$

- $x : X \quad \Rightarrow \quad Fx : Y$
- $F(\_,\_) : \int^{X^{\mathrm{op}} \times X} \mathrm{CAT}_{n-1}\left(X(x,y), Y(Fx,Fy)\right)$

The end is defined by universal property and encodes naturality.
Its elements can be constructed inductively

- $x, y : X \quad \Rightarrow \quad F(x,y) : X(x,y) \to Y(Fx,Fy)$
- $x_1, x_2, y_1, y_2 : X \quad \Rightarrow \quad F^{(2)}_{x_1,x_2,y_1,y_2} : \int^{f_x : X^{\mathrm{op}}(x_1,x_2), f_y : X(y_1,y_2)}$

$$\mathrm{CAT}_{n-1}\left(X(x_1,y_1), Y(Fx_2,Fy_2)\right)\left(F(x_2,y_2)X(f_x,f_y),\right.$$
$$\left. Y(F(x_1,x_2)f_x, F(y_1,y_2)f_y)F(x_1,x_2)\right)$$

The degree decreases at each step hence it stops at $F^{(n)}$.

Objective $X, Y \; : \; \mathrm{C{\scriptsize AT}}_n \;\Rightarrow\; \mathrm{C{\scriptsize AT}}_n(X, Y) \; : \; \mathrm{C{\scriptsize AT}}_n$

By Definition We need to show the following:

- $(\_ : \mathrm{C{\scriptsize AT}}_n(X, Y)) \; : \; \mathbb{V} \to \mathbb{B}$
- $\mathrm{C{\scriptsize AT}}_n(X, Y)(\_, \_) \; : \; \mathrm{C{\scriptsize AT}}_n(X, Y)^{\mathrm{op}} \times \mathrm{C{\scriptsize AT}}_n(X, Y) \to \mathrm{C{\scriptsize AT}}_{n-1}$

Second Step

- $F, G : \mathrm{C{\scriptsize AT}}_n(X, Y) \Rightarrow \mathrm{C{\scriptsize AT}}_n(X, Y)(F, G) : \mathrm{C{\scriptsize AT}}_{n-1}$

$$\mathrm{C{\scriptsize AT}}_n(X, Y)(F, G) := \int^{x:X} Y(Fx, Gx)$$

- $F_1, G_1, F_2, G_2 : \mathrm{C{\scriptsize AT}}_n(X, Y) \Rightarrow$

$$\mathrm{C{\scriptsize AT}}_n(X, Y)^{\mathrm{op}}(F_1, G_1) \times \mathrm{C{\scriptsize AT}}_n(X, Y)(F_2, G_2)$$
$$\to \mathrm{C{\scriptsize AT}}_n\left[\mathrm{C{\scriptsize AT}}_n(X, Y)(F_1, F_2), \mathrm{C{\scriptsize AT}}_n(X, Y)(G_1, G_2)\right]$$

Straightforward consequence of the structure of Y

Back to the Definition

$\mathrm{CAT}_n(\_,\_) : \mathrm{CAT}_n^{\mathrm{op}} \times \mathrm{CAT}_n \to \mathrm{CAT}_n$

Objective $\quad \mathrm{CAT}_n \quad : \quad \mathrm{CAT}_{n+1}$

By Definition We need to show the following:

- $(\_ : \mathrm{CAT}_n) \quad : \quad \mathbb{V} \to \mathbb{B}$
- $\mathrm{CAT}_n(\_,\_) \quad : \quad \mathrm{CAT}_n^{\mathrm{op}} \times \mathrm{CAT}_n \to \mathrm{CAT}_{n+1-1}$

Last Step

Promote to an $n+1$-functor
$X, Y : \mathrm{CAT}_n \Rightarrow \mathrm{CAT}_n(X, Y) : \mathrm{CAT}_n$

$$\mathrm{CAT}_n^{\mathrm{op}}(X_1, Y_1) \times \mathrm{CAT}_n(X_2, Y_2) \to \mathrm{CAT}_n(\mathrm{CAT}_n(X_1, X_2), \mathrm{CAT}_n(Y_1, Y_2))$$

$$f_1 : Y_1 \to X_1,\ f_2 : X_2 \xrightarrow{f_2} Y_2 \quad \Rightarrow \quad (X_1 \xrightarrow{f} X_2 \quad \Rightarrow \quad Y_1 \xrightarrow{f_1} X_1 \xrightarrow{f} X_2 \xrightarrow{f_2} Y_2)$$

$f_2 f f_1 :$

- $y : Y_1 \overset{f_1}{\Rightarrow} f_1 y : X_1 \overset{f}{\Rightarrow} f f_1 y : X_2 \overset{f_2}{\Rightarrow} f_2 f f_1 y : Y_2$
- $y, z : Y_1 \Rightarrow f_2 f f_1(x, y) : Y_1(y, z) \to Y_2(f_2 f f_1 y, f_2 f f_1 z)$

$$Y_1(y, z) \xrightarrow{f_1(y,z)} X_1(f_1 y, f_1 z) \xrightarrow{f(f_1 y, f_1 z)} X_2(f f_1 y, f f_1 z) \xrightarrow{f_2(f_2 f f_1 y, f_2 f f_1 z)} Y_2(f_2 f f_1 y, f_2 f f_1 z)$$

CAT_0

Posets and order preserving maps

An element of CAT_0 correspond to a poset:

$$P : \mathrm{CAT}_0 \Leftrightarrow \begin{array}{lccl} - & (\_ : P) & : & \mathbb{V} \to \mathbb{B} \\ - & P(\_,\_) & : & P^{\mathrm{op}} \times P \to \mathbb{B} \end{array}$$

- The naturality of the structural map corresponds to transitivity
- Maps between poset inherit a poset structure

Poset maps

$(P, Q : \mathrm{CAT}_0) \Rightarrow \mathrm{CAT}_0(P, Q) : \mathrm{CAT}_0$

Explicitly

- $$(f : P \to Q) := \begin{array}{ll} - & x : P \Rightarrow fx : Q \\ - & p, q : P \Rightarrow (f(p,q) : P(p,q) \Rightarrow Q(fp, fq)) \end{array}$$

- $(f, g : P \to Q) \Rightarrow \quad \mathrm{CAT}_0(P, Q)(f, g) := \int^{x:P} Q(fx, gx) : \mathbb{B}$

where $\alpha : \int^{x:P} Q(fx, gx) \Leftrightarrow (x : P \Rightarrow \alpha_x : Q(fx, gx))$

Example

The Poset of Natural Numbers

We let $\_ : \mathbb{N} : \mathbb{V} \to \mathbb{B}$ be defined as

- $(0 : \mathbb{N}) := \top$,
- $(x : \mathbb{N} \Rightarrow x + 1 : \mathbb{N})$

We let $\mathbb{N}(\_, \_) : \mathbb{N}^{\mathrm{op}} \times \mathbb{N} \to \mathbb{B}$ be induced by

- $\mathbb{N}(x, x + 1) := \top$,

Sets

We define a **set** as a **poset with additional structure**.

$$(X : \textsc{Set}) := (X : \textsc{Cat}_0) \times (X \xrightarrow{\simeq} X^{\mathrm{op}})$$

which means $x, y : X \Rightarrow (X(x, y) \Leftrightarrow X(y, x))$. We also write $x = y$

Maps

A map $f : X \to Y$ between sets is given by

- $x : X \Rightarrow fx : Y$
- $x, y : X \Rightarrow (x = y \Rightarrow fx = fy)$

The Set of Maps

Suppose $f, g : X \to Y$ are maps between sets. Equality is defined as

$$(f = g) := \int^{x:X} (fx = fy) : \mathbb{B}$$

Sets

The Poset of Subsets of a Set

The structural map of a set X yields

$$X \to \textsc{Cat}_0(X^{\mathrm{op}}, \mathbb{B}) =: \mathcal{P}X$$

The Category of Sets

Satisfies all axioms of ETCS.

Quotients

Are very easy to define: suppose $R(\_,\_) : X^{\mathrm{op}} \times X \to \mathbb{B}$.

- Same objects: $(\_ : X/R) := (\_ : X)$
- Define equality: $X/R(x,y) := R(x,y)$

A map $X/R \to Y$ precisely corresponds to a map $f : X \to Y$ such that $R(x,y) \Rightarrow fx = fy$.

Equivalences

A Recursive Definition

Let $F : X \to Y$ be a map in $\mathrm{C{\scriptsize AT}}_n$. We say that

- F is a 0-equivalence if

$$(y : Y \Rightarrow (F^* y : X) \times (e_y : Y(FF^* y, y) \times Y(y, FF^* y))$$

- F is an r-equivalence if F is a 0-equivalence and (F, F^*, e) yields an $r - 1$ equivalence

$$x, y : X \Rightarrow F(x, y) : X(x, y) \to Y(Fx, Fy)$$

which is natural in x, y.

$$X(x, y) \xrightarrow{\quad F(x,y) \quad} Y(Fx, Fy)$$

$$F^* Fx \xrightarrow{F^* f} F^* Fy$$

$$\begin{array}{ccc} Fx & \xrightarrow{\quad f \quad} & Fy \\ {\scriptstyle e_{Fx}} \downarrow & & \downarrow {\scriptstyle e_{Fy}} \\ FF^* Fx & \xrightarrow[FF^* f]{} & FF^* Fy \end{array}$$

Equivalences

Let $F : X \to Y$ be a map in $\mathrm{C}\small{\mathrm{AT}}_n$.

- We say that F is an equivalence is it is an $n+1$-equivalence.
- We obtain an n-category of equivalences $\mathrm{C}\small{\mathrm{AT}}_n(X, Y)^{\simeq}$

In Sets

- 0-equivalences correspond to surjective maps
- 1-equivalences correspond to isomorphisms

Axiom of Choice

- Does not hold: consider a quotient map $\pi : X \to X/R$.
- It is 0-equivalence: $x : X/R \Leftrightarrow x : X$.
- It does not satisfy $X/R(x, y) = R(x, y) \Rightarrow X(x, y)$ unless it is trivial

In Cat

- 0-equivalences: essentially surjective functors.
- 1-equivalences: full and essentially surjective.
- 2-equivalences: fully faithful and essentially surjective.

Identity, Undiscernability and Equality

The Shadow of Sets

Philosophy

Equality makes sens within a set.
Objects of a category can only be distinguished up to isomorphism.
Objects of an n-category can only be distinguished up to equivalence.

How to define sameness ?

If I define $X :=$ *expression*
then I want X and *expression* to refer to the same thing
in a sense that is stronger than equivalence.

Literal Equality

Literal symbols form a set. We regard a definition as a process that introduces a literal equality between X and *expression*.

In Type Theory

Literal equality corresponds to the definitional equality.

Interdimensional Connexions

The Truncation - Inclusion Adjunction

in $\textsc{Cat}_{n+1}$

$$\Pi : \textsc{Cat}_n \leftrightarrows I\textsc{Cat}_{n-1} : I$$

$X : \textsc{Cat}_n \Rightarrow \Pi X : \textsc{Cat}_{n-1}$

- $\pi x : \Pi X \Leftrightarrow x : X$
- $x, y : X \Rightarrow \Pi X(x, y) := \Pi\left(X(x, y)\right) : \textsc{Cat}_{n-2}$

$X : \textsc{Set} \Rightarrow \Pi X : \mathbb{B}$

Is false if X is empty and true else.

X:$\textsc{Cat}_n \Rightarrow IX : \textsc{Cat}_{n+1}$

- $\iota x : IX \Leftrightarrow x : X$
- $x, y : X \Rightarrow IX(\iota x, \iota y) := I\left(X(x, y)\right) : \textsc{Cat}_{n-2}$

$\tau : \mathbb{B} \Rightarrow I\tau : \textsc{Set}$

$I\bot = \varnothing$ and $I\top = *$

Small Objects

An object $X : \mathrm{CAT}_n$ is r-**small** if it is equipped with

- An object $X_r : \mathrm{CAT}_r$
- An r-equivalence $I^{n-r} X_r \to X$

We write $X : \mathrm{CAT}_n^{(r)}$ for an r-small objects in CAT_n.

- We have a notion of r-small functors between r-small objects.
- r-small objects, together with r-**small functors**, form an $r+1$-small n-category $\mathrm{CAT}_n^{(r)} : \mathrm{CAT}_n^{(r+1)}$
- Hence $\mathrm{CAT}_n^{(r)}$ has an underlying $r+1$-category $Cat_n^{(r)} : \mathrm{CAT}_{r+1}$

Small objects in CAT

- 0-small categories are categories equipped with an equality relation on the objects - underlying set of objects.
- 0-small functors are strict functors.
- We obtain a category of small categories and strict functors.

$\mathrm{CAT}_\omega : \mathrm{CAT}_{\omega+1}$

We say that X is an ω-category and we write $X : \mathrm{CAT}_\omega$ if

$n : \mathbb{Z} \quad \Rightarrow$
- $X_{(n)} : \mathrm{CAT}_n$
- $\sqcap_n^X : X_{(n+1)} \rightleftarrows I_{n+1} X_{(n)} : I_{n+1}^X$

Theorem

$$\mathrm{CAT}_\omega : \mathrm{CAT}_{\omega+1}$$

Idea

- $(\mathrm{CAT}_\omega)_{(n)} := \mathrm{CAT}_n : \mathrm{CAT}_{n+1}$
- $\sqcap : \mathrm{CAT}_{n+1} \leftrightarrows I\mathrm{CAT}_n : I$

Remark

$$I\mathrm{CAT}_\omega \simeq \mathrm{CAT}_{\omega+1}$$

Homotopy Type Theory

Interpretation

$$\mathbb{J}_{\mathrm{HoTT}} \longrightarrow \mathbb{B}$$

$$X : \mathcal{U}_n \mapsto X : \mathrm{CAT}^{\simeq}_{\omega+n}$$

$$X : \mathcal{U}_n \vdash x : X \mapsto (X : \mathrm{CAT}^{\simeq}_{\omega+n}, x : X)$$

$$(x \equiv y) \mapsto (x =_{\mathbb{L}} y)$$

$$\mathbb{J}_{\mathrm{HoTT}}(J_1, \dots, J_r, J) \to \mathbb{B}([J_1] \times \cdots \times [J_r], [J])$$

$\mathcal{U}$–INTRO $\quad \dfrac{\Gamma \text{ ctx}}{\Gamma \vdash \mathcal{U}_n : \mathcal{U}_{n+1}} \mapsto (\mathrm{CAT}^{\simeq}_{\omega+n} : \mathrm{CAT}^{\simeq}_{\omega+n+1})$

$\mathcal{U}$–CUMUL $\quad \dfrac{\Gamma \vdash A : \mathcal{U}_n}{\Gamma \vdash A : \mathcal{U}_{n+1}} \mapsto (A : \mathrm{CAT}^{\simeq}_{\omega+n} \Rightarrow IA : \mathrm{CAT}^{\simeq}_{\omega+n+1})$

Π–FORM $\quad \dfrac{\Gamma \vdash A : \mathcal{U}_n \quad \Gamma, x : A \vdash B : \mathcal{U}_n}{\Gamma \vdash \prod_{x:A} B : \mathcal{U}_n} \mapsto$

$$A : \mathrm{CAT}^{\simeq}_{\omega+n}, B : IA \to \mathrm{CAT}^{\simeq}_{\omega+n}$$
$$\Rightarrow \prod_{x:A} Bx := [IA, \mathrm{CAT}_{\omega+n}](*, B)$$
$$\simeq \int^{x:A} Bx : \mathrm{CAT}^{\simeq}_{\omega+n}$$

Going further

Generalized truth values

Idea

Work with a different cartesian closed poset for truth values.

Example

Let $\mathbb{B}^+$ with

$$\bot \Rightarrow + \Rightarrow \top$$

Maybe is that which may have proof. The arrows which are written correspond to the elements in the morphisms:

- $\mathbb{B}^+(\bot, +) = \top$
- $\mathbb{B}^+(+, \top) = \top$
- $\mathbb{B}^+(\top, +) = +$
- $\mathbb{B}^+(+, \bot) = +$
- $\mathbb{B}^+(\top, \bot) = \bot$
- $\mathbb{B}^+(+, \bot) = \bot$

The arrows that may be can be represented as dotted arrows:

$$\top \dashrightarrow + \dashrightarrow \bot$$

- Objects at level 0 in the corresponding structure have potential elements that may or may not

We can work formally within a hierarchy of n-types $\mathbb{T}_n : \mathbb{T}_{n+1}$

$$\text{satisfying } (X : \mathbb{T}_n) := \begin{array}{lrcl} - & (\_ : X) & : & \mathbb{V} \to \mathbb{B} \\ - & X(\_,\_) & : & X^{\mathrm{op}} \times X \to \mathbb{T}_{n-1} \end{array}$$

Example

$\mathrm{CAT}_n^{\mathbb{N}} : \mathrm{CAT}_{n+1}^{\mathbb{N}}$

$X : \mathrm{CAT}_n^{\mathbb{N}} \Rightarrow X(\_,\_) : X^{\mathrm{op}} \times X \to \mathrm{CAT}_{n-1}^{\mathbb{N}}$ in $\mathrm{CAT}_{n-1}^{\mathbb{N}}$

given in n by $X(n)(\_,\_) : X(n)^{\mathrm{op}} \times X(n) \to \mathrm{CAT}_{n-1}$

Monoidal objects can be defined provided that they live in a bigger object that is itself equipped with a monoidal structure.

Monoidality

$$(\textsc{Cat}_n, \times_n) : \textsc{Mon}_{(\textsc{Cat}_{n+1}, \times_{n+1})}$$
$$\textsc{Mon}_{(\textsc{Cat}_{n+1}, \times_{n+1})}\,(*_n, \textsc{Cat}_n) \simeq \textsc{Mon}_{(\textsc{Cat}_n, \times_n)}$$

Operads and Algebras

$$\left(\mathbb{T}_n, \otimes_n^{\mathbb{P}}\right) : \mathbb{P}\textsc{Alg}_{\left(\mathbb{T}_{n+1}, \otimes_{n+1}^{\mathbb{P}}\right)}$$

Key Features

This framework is very easy to work with.

- Compact.
- Strongly suited for inductive reasonning.
- Processes literal expressions directly into truth values.
- Avoids the final process of interpretation of the judgments as boolean truth values hence enables much more general logics.
- Provides more general notions of topos and higher structures.
- Encapsulates the nested nature of structures well. Strongly suited for higher structures.
- Is both computational and has an homotopical behaviour.
- The level of freedom in potential generalisations is high.
- It reflects the actual way we do mathematics.

Thanks

- **Any remark:** sophiedespalungue@gmail.com
- **Source:** d'Espalungue d'Arros, S. (2023). *Operads in 2-categories and models of structure interchange*. https://theses.hal.science/tel-04617115
- **Further:**
  - Formal Category Theory: Chapter 1
  - Formal Operad Theory: Chapter 2
  - Structure Interchange: Chapter 3
  - Foundations: Appendix